Temperature of the inflaton and duration of inflation from WMAP data

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If the initial state of the inflaton field is taken to have a thermal distribution instead of the conventional zero particle vacuum state then the curvature power spectrum gets modified by a temperature dependent factor such that the fluctuation spectrum of the microwave background radiation is enhanced at larger angles. We compare this modified cosmic microwave background spectrum with Wilkinson microwave anisotropy probe data to obtain an upper bound on the temperature of the inflaton at the time our current horizon crossed the horizon during inflation. We further conclude that there must be additional e-foldings of inflation beyond what is needed to solve the horizon problem.

The inflationary paradigm [1] provides a ready solution for the horizon and flatness problems while simultaneously providing a source for the initial density perturbations required as a seed for the large scale structure in the universe today. During inflation the energy density of the universe is dominated by the potential energy of a scalar field leading to a brief period of accelerated expansion.

The amplitude for quantum fluctuations in the inflaton field ϕ generated during inflation effectively freezes out for a particular comoving scale once that mode crosses the horizon. These fluctuations also translate into perturbations in energy density and curvature in the universe. In the inflationary paradigm these perturbations are the source of the observed anisotropy in the microwave background and the seed for large scale structure. Conventionally the fluctuation power spectrum for the inflaton is calculated using the zero particle vacuum. However if inflation was preceded by a radiation era then the inflaton was in thermal equilibrium at some point in the early universe, via at least gravitational couplings at the Planck scale. In chaotic [2] and natural inflation models with a symmetry breaking scale $f \sim M_{Pl}$ [3], non-zero momentum modes can be in thermal equilibrium due to gravitational interactions with the existing radiation. The inflaton (or any other field which contributes to the density perturbation during inflation) can then have a thermal distribution corresponding to the temperature $\mathcal{T}_i$ at the beginning of inflation. Though the inflaton field may be decoupled from the rest of radiation before inflation it retains its thermal distribution in an adiabatically expanding universe. (This is different from the warm inflation scenarios [4] where the inflaton decays to radiation during inflation.) In reference [5] it was argued that the power spectrum of gravitational waves generated during inflation has an extra temperature dependent factor due to the thermal gravitons which decouple at the Planck era and retain their thermal distribution.

In this Letter we include the initial thermal nature of the inflaton ϕ in the generation of the inflaton fluctuation power spectrum. The power spectrum of the perturbations of the inflaton, and hence of the curvature, has an extra temperature dependent factor $\coth[k/(2a_i\mathcal{T}_i)]$ where k is the wavenumber of the modes in comoving coordinates and a_i is the scale factor at t_i when inflation commences. The physical wavenumber is k/a and physical temperature $\mathcal{T} = T/a$, where T is the comoving temperature during inflation. Using this thermal power spectrum we calculate the CMB anisotropy using CMBFAST [6]. We compare the result with data from WMAP [7] and find that the comoving temperature of the inflaton is constrained to be $T < 1.0 \times 10^{-3}\,\mathrm{Mpc}^{-1}$. Since $T = \mathcal{T}_0 a_0$, where $\mathcal{T}_0$ and a_0 are the temperature and scale factor when our current horizon scale crossed the horizon during inflation, this constraint can be rewritten as $\mathcal{T}_0 < 4.2H$, where H is the Hubble parameter during inflation (all bounds stated in this paper are at 1σ or $68\% C.L.$). This result is valid independent of the scale of inflation. If inflation takes place in the GUT era ($M_{GUT} \sim 10^{15}\,\mathrm{GeV}$) then $\mathcal{T}_0 < 1.0 \times 10^{12}\,\mathrm{GeV}$ and for the inflation at the electroweak era ($M_{EW} \sim 10^{3}\,\mathrm{GeV}$) [8] we have the bound $\mathcal{T}_0 < 1.0 \times 10^{-12}\,\mathrm{GeV}$. Since $\mathcal{T}_0$ is much less than the energy scale of inflation, M_{inf}, the duration of inflation must be longer than what is needed to solve the horizon problem in models where the temperature at the beginning of inflation $\mathcal{T}_i \sim M_{inf}/7$, the energy scale of inflation. If ΔN is the number of e-foldings from the beginning of inflation to the time our current horizon scale crossed the de Sitter horizon, then the bound on $\mathcal{T}_0$ from WMAP data implies $\Delta N = \ln(\mathcal{T}_i/\mathcal{T}_0) > \ln(0.03\,M_{inf}/H)$. For GUT scale ($M_{inf} = 10^{15}\,\mathrm{GeV}$) inflation $\Delta N_{GUT} > 7$ and for inflation at the the electroweak scale $\Delta N_{EW} > 32$.

We now calculate the curvature power spectrum due to inflaton fluctuations. Presuming a quasi de Sitter universe during inflation, conformal time τ ($d\tau \equiv dt/a$) and the scale factor during inflation $a(\tau)$ are related by $a(\tau) = -1/H\tau(1-\epsilon)$ where $\epsilon = -\frac{\dot{H}}{H^2} = \frac{4\pi}{M_{Pl}^2}\frac{\dot{\phi}^2}{H^2} = \frac{M_{Pl}^2}{16\pi}\left(\frac{V'}{V}\right)^2$, V is the inflaton potential, V' stands for the derivative of the potential with respect to ϕ and $\dot{H}$ implies a derivative of H with respect to cosmic time. Henceforth a dot will always signify a derivative with respect to cosmic time whereas a prime may mean a derivative with respect to

the conformal time when it appears with quantities which are explicit functions of time or a derivative with respect to the inflaton field when it appears with the potential. M_{Pl} is the Plank mass. We define additional slow roll parameters as $\eta = \frac{M_{Pl}^2}{8\pi}\left(\frac{V''}{V}\right)$ and $\delta = \ddot{\phi}/(H\dot{\phi}) = \epsilon - \eta$. We further define $z = \frac{a\dot{\phi}}{H}$ and $u = a(\tau)\delta\phi(\mathbf{x},\tau)$, where $\delta\phi(\mathbf{x},\tau)$ is the inflaton field perturbations on spatially flat hypersurfaces. Then the action for u is given as [9–11]:

$$S = \frac{1}{2}\int d\tau\, d^3x \left[(u')^2 - (\nabla u)^2 + \frac{z''}{z}u^2\right], \tag{1}$$

The gauge invariant comoving curvature perturbation is defined as $\mathcal{R} = \psi - H\frac{\delta\phi}{\dot{\phi}}$ where ψ is the usual scalar metric perturbation. Consequently on spatially flat hypersurfaces $u = -z\mathcal{R}$. Expressing $u(\mathbf{x},\tau)$ as a quantum field we can write:

$$\hat{u}(\mathbf{x},\tau) = \int \frac{d^3k}{(2\pi)^{3/2}}\left(a_{\mathbf{k}}\; f_k(\tau) + a^{\dagger}_{-\mathbf{k}}\; f^*_k(\tau)\right) e^{i\mathbf{k}\cdot\mathbf{x}}, \tag{2}$$

where $\mathbf{k}$ is the comoving wavenumber.

The gauge invariant comoving curvature perturbation can then be expressed as:

$$\begin{aligned}\mathcal{R} &= \frac{-1}{z}\int \frac{d^3k}{(2\pi)^{3/2}}\left(a_{\mathbf{k}}\; f_k(\tau) + a^{\dagger}_{-\mathbf{k}}\; f^*_k(\tau)\right) e^{i\mathbf{k}\cdot\mathbf{x}},\\ &\equiv \int \frac{d^3k}{(2\pi)^{3/2}}\mathcal{R}_{\mathbf{k}}(\tau)\, e^{i\mathbf{k}\cdot\mathbf{x}}.\end{aligned} \tag{3}$$

The power spectrum of the comoving curvature perturbations can be defined by the relation

$$\langle \mathcal{R}_{\mathbf{k}}\mathcal{R}^*_{\mathbf{k}}\rangle \equiv \frac{2\pi^2}{k^3}P_{\mathcal{R}}\,\delta^3(\mathbf{k}-\mathbf{k}') \tag{4}$$

The usual quantization condition between the fields and their canonical momenta yields $[a_{\mathbf{k}}, a^{\dagger}_{\mathbf{k}'}] = \delta^3(\mathbf{k}-\mathbf{k}')$ and the vacuum satisfies $a_{\mathbf{k}}|0\rangle = 0$. If the inflaton field had zero occupation prior to inflation then $\langle a^{\dagger}_{\mathbf{k}}a_{\mathbf{k}}\rangle = 0$ and we would obtain a correlation function $\sim |f_k(\tau)|^2$. However if the inflaton field was in thermal equilibrium at some earlier epoch [12] it will retain its thermal distribution even after decoupling from the other radiation fields and its occupation number will be given by:

$$\langle a^{\dagger}_{\mathbf{k}}a_{\mathbf{k}'}\rangle = \frac{1}{e^{E_k/\mathcal{T}_f}-1}\delta^3(\mathbf{k}-\mathbf{k}'), \tag{5}$$

where E_k is the energy corresponding to the k mode at the inflaton decoupling temperature $\mathcal{T}_f$. For effectively free modes $E_k \approx k/a_f$, and so $E_k/\mathcal{T}_f = k/(a_f\mathcal{T}_f) = k/(a_i\mathcal{T}_i) = k/T$.

Using Eq. (3) and Eq. (5) it can be seen that

$$\begin{aligned}\langle \mathcal{R}_{\mathbf{k}}\mathcal{R}^*_{\mathbf{k}}\rangle &= \frac{1}{|z|^2}\left(1 + \frac{2}{e^{\frac{k}{T}}-1}\right)|f_k(\tau)|^2\,\delta^3(\mathbf{k}-\mathbf{k}')\\ &= \frac{1}{|z|^2}|f_k(\tau)|^2\,\coth\left[\frac{k}{2T}\right]\,\delta^3(\mathbf{k}-\mathbf{k}')\end{aligned} \tag{6}$$

From the defining relation Eq. (4) for the curvature power spectrum and Eq. (6) we find that the power spectrum for the curvature perturbations can be expressed in terms of the mode functions $f_k(\tau)$ as

$$P_{\mathcal{R}}(k) = \frac{k^3}{2\pi^2}\frac{|f_k|^2}{|z|^2}\,\coth\left[\frac{k}{2T}\right] \tag{7}$$

For constant ϵ and δ the mode functions $f_k(\tau)$ obey the minimally coupled Klein-Gordon equation [11,13],

$$f''_k + \left[k^2 - \frac{1}{\tau^2}\left(\nu^2 - \frac{1}{4}\right)\right]f_k = 0, \tag{8}$$

where $k = |\mathbf{k}|$ and, for small ϵ and δ, $\nu = \frac{3}{2} + 2\epsilon + \delta$. Eq. (8) has the general solution given by,

$$f_k(\tau) = \sqrt{-\tau}\left[c_1(k)\,H^{(1)}_{\nu}(-k\tau) + c_2(k)\,H^{(2)}_{\nu}(-k\tau)\right]. \tag{9}$$

When the modes are well within the horizon they can be approximated by flat spacetime solutions ${f_k}^0(\tau) = \frac{1}{\sqrt{2k}}e^{-ik\tau}, \quad (k \gg aH)$. Matching the general solution in Eq. (9) with the solution in the high frequency ("flat spacetime") limit gives the value of the constants of integration $c_1(k) = \frac{\sqrt{\pi}}{2}e^{i(\nu+\frac{1}{2})\frac{\pi}{2}}$ and $c_2(k) = 0$. Eq. (9) then implies that for $-k\tau \gg 1$ or $k \ll aH$,

$$f_k(\tau) = e^{i(\nu-\frac{1}{2})\frac{\pi}{2}}2^{\nu-\frac{3}{2}}\frac{\Gamma(\nu)}{\Gamma(\frac{3}{2})}\frac{1}{\sqrt{2k}}(-k\tau)^{\frac{1}{2}-\nu}. \tag{10}$$

Substituting the solution in Eq. (10) for the the super-horizon modes ($k \ll aH$) in the expression Eq. (7) for the curvature power spectrum we obtain

$$P_{\mathcal{R}}(k) = \frac{H^4}{4\pi^2\dot{\phi}^2}\left(\frac{k}{aH}\right)^{n_s-1}\coth\left[\frac{k}{2T}\right] \tag{11}$$

with $n_s - 1 = 3 - 2\nu = -4\epsilon - 2\delta$. Note that since $f_k(\tau) \sim z(\tau)$ for superhorizon modes, $P_{\mathcal{R}}(k)$ is time independent.

We can now rewrite the power spectrum as,

$$P_{\mathcal{R}}(k) = A(k_0)\left(\frac{k}{k_0}\right)^{n_s-1}\coth\left[\frac{k}{2T}\right] \tag{12}$$

where k_0 is referred to as the pivot point and $A(k_0)$ is the normalisation constant.

$$A(k_0) = \frac{H^4}{4\pi^2\dot{\phi}^2}\Big|_{aH=k_0} = \frac{H^2_{k_0}}{\pi M^2_{Pl}\epsilon}, \tag{13}$$

where H_{k_0} is the Hubble parameter evaluated when $aH = k_0$ during inflation.

During inflation a mode k leaves the horizon when $k/a = H$. Therefore for any k the initial thermal effects modify the scale-free power spectrum by $\coth(a_k H_k/2T) = \coth(H_k/2\mathcal{T}_k)$, where a_k and $\mathcal{T}_k$ are the scale factor and temperature when the mode k leaves the horizon.

We now use the curvature power spectrum as given in Eq. (12) to obtain the CMB fluctuations. We make use of the publicly available code CMBFAST [6] and modify the power spectrum formula, according to our need, in subroutine powersflat in the program cmbflat.F. We set the parameters $\Omega_b = 0.046, \Omega_m = 0.27, h = 71.0$ as determined in [14] for a flat universe with no running of the scalar index n_s. We set $k_0 = 0.05\,\mathrm{Mpc}^{-1}$.

Fig. 1 shows two different possible CMB TT-correlation curves for two comoving inflaton temperatures, with $n_s = 0.99$ and $A(k_0) = 2.01 \times 10^{-9}$. It shows clearly that as the comoving temperature of the inflatons decreases we get a better fit of our TT-correlations with the WMAP data. The curves show that the change of temperature T of the inflatons essentially affects the lower multipoles or higher length scales of the TT-correlations while the higher multipoles remains unaffected. This is expected because $\coth(x) \sim 1$ for $x \gg 1$.

We now do a 3-parameter analysis by varying $A(k_0)$ and T for multiple values of n_s from $n_s = 0.90$ to $n_s = 1.1$, and each time we compare the resultant TT-anisotropy with the WMAP data [7] and calculated the χ^2 of the resultant computed distribution with respect to the WMAP data. From this three parameter fit we find that χ^2 minimum occurs at the values $A(k_0) = 2.0\times10^{-9}$, $n_s = 0.99$ and at the comoving temperature of the inflaton field $T = 0$. In Fig. 2 we show the 1σ allowed regions of T and $A(k_0)$ at different values of n_s. The regions inside the curves are allowed by WMAP data, with the best fit value being $A(k_0) = 2.0 \times 10^{-9}$, $n_s = 0.99$, and comoving temperature $T = 0$. The maximum allowed values for T increases when we increase n_s from $n_s = 0.90$ to $n_s = 1.06$. For values of n_s greater than 1.06 maximum allowed value of T decreases monotonically. So from this three parameter fits we see that the upper bound on the comoving temperature of the inflaton field is $T < 0.001\,\mathrm{Mpc}^{-1}$.

The power spectrum of tensor perturbations also has an identical $\coth(k/2T)$ factor due to the thermal spectrum of the gravitons in the initial state [5]. The tensor-to-scalar amplitude ratio $r = 16\epsilon < 1.28\,(95\%C.L.)$ [15]. In our parameter fits we do not violate this upper bound on ϵ which is independent of temperature.

The temperature of the inflaton ($\mathcal{T}_0$) when our present horizon was leaving the de Sitter horizon (H^{-1}) can be calculated, $T = \mathcal{T}_0 a_0 < 4.2 R_h^{-1}$ (where a_0 is the scale factor at the time of this horizon crossing). We then obtain

$$\mathcal{T}_0 < 4.2 \left(\frac{a_{now}}{a_0}\right) R_h^{-1} = 4.2H\,, \tag{14}$$

where a_{now} is the current scale factor and $R_h/a_{now} = H^{-1}/a_0$. $H = (8\pi/3)^{1/2} M_{inf}^2/M_{Pl}$, ignoring the variation in H during inflation. The constraint Eq. (14) is valid whatever be the scale of inflation. If inflation takes place in the GUT era ($M_{GUT} \sim 10^{15}\,\mathrm{GeV}$) then $\mathcal{T}_0 < 1.0 \times 10^{12}\,\mathrm{GeV}$, and for inflation at the electroweak scale ($M_{EW} \sim 10^3\,\mathrm{GeV}$) we have the bound $\mathcal{T}_0 < 1.0 \times 10^{-12}\,\mathrm{GeV}$. In general, for $M_{inf} < 1.4 \times 10^{17}\,\mathrm{GeV}$, $\mathcal{T}_0 < \mathcal{T}_i$.

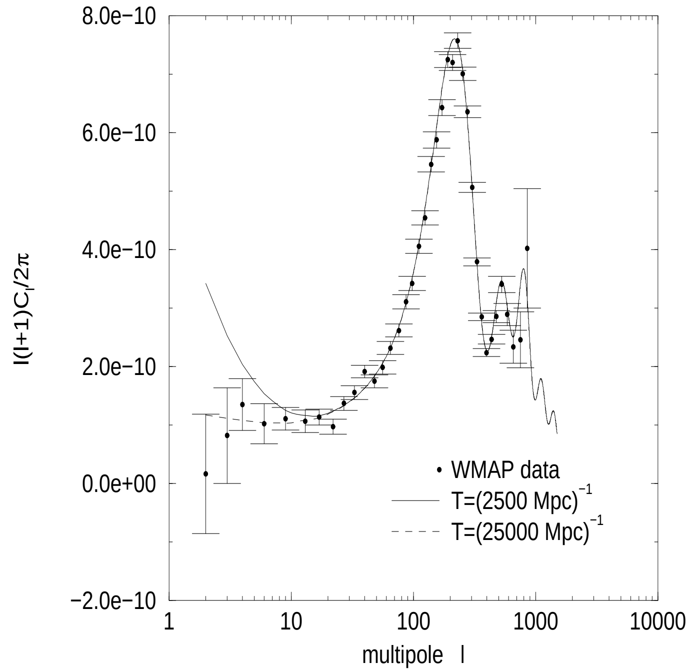

FIG. 1. The dotted points with the error bars corresponds to the WMAP binned data for the CMB TT-anisotropies. The solid and dashed curves are generated from CMBFAST with the inflaton comoving temperature $T = (2500\mathrm{Mpc})^{-1}$ and $T = (25000\mathrm{Mpc})^{-1}$ respectively. The error bars on the WMAP data consists of measurement errors and errors attributed to cosmic variance.

From Fig. 2 we see that T has an upper bound at 1σ given by

$$T < 1.0 \times 10^{-3}\,\mathrm{Mpc}^{-1} = 4.2 R_h^{-1}\ (68.3\%C.L.) \tag{15}$$

where we have expressed the bound on the comoving temperature in terms of the present horizon scale $R_h \simeq 4200\,\mathrm{Mpc}$.

The minimum number of e-foldings of inflation required to solve the horizon problem is obtained by taking our current horizon scale to be the first to cross the horizon during inflation. If there is a thermal background of the inflaton and inflation commences at $\mathcal{T}_i \sim M_{inf}/7$ our results above imply that our current horizon must have

crossed the de Sitter horizon some time after the beginning of inflation. Thus the duration of inflation must be longer than what is just needed to solve the horizon problem by $\Delta N = \ln(\mathcal{T}_i/\mathcal{T}_0)$.

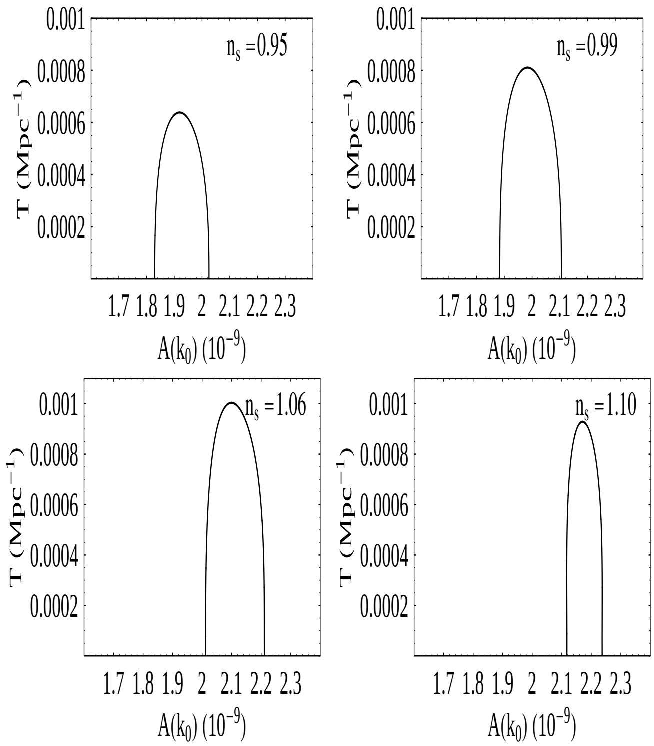

FIG. 2. Values of T and $A(k_0)$ allowed at 1σ by WMAP data are the regions inside the curves. The plots at different values of n_s show that the maximum allowed value of T is $0.001\,\mathrm{Mpc}^{-1}$.

The upper bound on $\mathcal{T}_0$ from WMAP then implies

$$\Delta N > \ln\left[0.03\frac{M_{inf}}{H}\right] = \ln\left[0.01\frac{M_{Pl}}{M_{inf}}\right] . \qquad (16)$$

Perturbations generated during these additional e-foldings are yet to enter our present horizon. For GUT scale inflation WMAP data implies $\Delta N_{GUT} > 7$ and for inflation at the the electroweak scale it implies $\Delta N_{EW} > 32$. For Planck scale inflation, such as chaotic and natural inflation, no additional e-foldings of inflation are required.

There exists a Lyth bound [16,17] on the minimum variation in the inflaton field during inflation, namely, $\delta\phi > M_{Pl}\sqrt{r/7}\,\delta N$, where δN is taken to be the number of e-foldings of inflation (~ 4.6) over which modes corresponding to multipoles $l < 100$ leave the horizon. This bound is further strengthened by a factor of 2.5 by including the additional 7 e-foldings (for GUT scale inflation) required by our analysis. The revision of the Lyth analysis by including the entire duration of inflation [18] should also include the additional e-foldings given in Eq. (16).

In conclusion, we find that if inflation is preceded by a hot radiation era then there will be a stimulated emission of inflaton perturbations during inflation into the initial thermal bath of the inflaton which will change the scale free spectrum. The change in the spectrum is large at large angles and there is no change in the CMB anisotropies at small angles. (This differs from the warm inflation scenario where the departure from scale invariance is proportional to the derivative of the inflaton dissipative term, which is small to meet the slow roll condition [19].) Furthermore, from the WMAP data we find that the minimum number of e-foldings of inflation is larger than that required to solve the horizon problem, with the increase varying from 7-32 for the energy scale of inflation varying from the GUT to the electroweak scale.

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